

# Affine $su(3)$ and $su(4)$ fusion multiplicities as polytope volumes

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## Abstract

Affine  $su(3)$  and  $su(4)$  fusion multiplicities are characterised as discretised volumes of certain convex polytopes. The volumes are measured explicitly, resulting in multiple sum formulas. These are the first polytope-volume formulas for higher-rank fusion multiplicities. The associated threshold levels are also discussed. For any simple Lie algebra we derive an upper bound on the threshold levels using a refined version of the Gepner-Witten depth rule.

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# 1 Introduction

We are interested in describing fusions of integrable highest weight modules of affine Lie algebras. Let  $M_\lambda$  denote an integrable highest weight module of an untwisted affine Lie algebra. The affine weight is uniquely specified by the highest weight  $\lambda$  of the simple horizontal subalgebra (the underlying Lie algebra), and the affine level  $k$ . Fusion of two such modules may be written as

$$M_\lambda \times M_\mu = \sum_{\nu} N_{\lambda,\mu}^{(k)\nu} M_\nu \quad , \quad (1)$$

where  $N_{\lambda,\mu}^{(k)\nu}$  is the fusion multiplicity. This is equivalent to studying the more symmetric problem of determining the multiplicity of the singlet in the expansion of the triple fusion

$$M_\lambda \times M_\mu \times M_\nu \supset N_{\lambda,\mu,\nu}^{(k)} M_0 \quad . \quad (2)$$

If  $\nu^+$  denotes the weight conjugate to  $\nu$ , we have  $N_{\lambda,\mu,\nu}^{(k)} = N_{\lambda,\mu}^{(k)\nu^+}$ .

The associated and level-independent tensor product multiplicity is denoted  $T_{\lambda,\mu,\nu}$ :

$$M_\lambda \otimes M_\mu \otimes M_\nu \supset T_{\lambda,\mu,\nu} M_0 \quad . \quad (3)$$

It is related to the fusion multiplicity as

$$T_{\lambda,\mu,\nu} = \lim_{k \rightarrow \infty} N_{\lambda,\mu,\nu}^{(k)} \quad . \quad (4)$$

It has been conjectured [1] that the fusion multiplicities are uniquely determined from the tensor product multiplicities and the associated multi-set of minimum levels  $\{t\}$  at which the various couplings first appear. Therefore, to the triplet  $(\lambda, \mu, \nu)$  there correspond  $T_{\lambda,\mu,\nu}$  distinct couplings, hence  $T_{\lambda,\mu,\nu}$  values of  $t$ , one for each distinct coupling. These values are called threshold levels. The threshold levels associated to two different couplings may be identical. That justifies the use of the notion multi-set of threshold levels to describe the general case. The number of different couplings with the same threshold level  $t$  is called the threshold multiplicity,  $n_{\lambda,\mu,\nu}^{(t)}$ , and may be expressed in terms of the fusion multiplicities:

$$n_{\lambda,\mu,\nu}^{(t)} = N_{\lambda,\mu,\nu}^{(t)} - N_{\lambda,\mu,\nu}^{(t-1)} \quad . \quad (5)$$

In Ref. [2] Berenstein and Zelevinsky showed that an  $su(r+1)$  tensor product multiplicity is equal to the number of possible triangular arrangements of non-negative integers subject to certain constraints, referred to as BZ triangles. A triangle without the constraint that all the integer entries should be *non-negative*, is called a generalised BZ triangle [3]. The set of generalised triangles associated to a particular product (3) spans an  $\frac{1}{2}r(r-1)$ -dimensional lattice. Any triangle  $\mathcal{T}$  in the lattice may be expressed in terms of an initial one  $\mathcal{T}_0$  plus an integer linear combination of so-called (basis) virtual triangles  $\mathcal{V}_l$ :

$$\mathcal{T} = \mathcal{T}_0 + \sum_{l=1}^{r(r-1)/2} v_l \mathcal{V}_l \quad . \quad (6)$$

Re-imposing the constraint that all entries must be non-negative, results in a set of inequalities in the coefficients  $v_l$  defining a convex polytope. Its discretised volume (i.e., the number of integer points enclosed by it) is the tensor product multiplicity [3].

This idea was generalised to higher-point couplings in [4], and to affine  $su(2)$  fusions in [5] (that work also describes the extension to higher-point fusions and higher genus). The objective of the present work is the extension to affine  $su(3)$  and  $su(4)$  fusions. Thus, the affine  $su(3)$  and  $su(4)$  fusion multiplicities are characterised as discretised volumes of certain polytopes. The volumes are subsequently measured explicitly, resulting in multiple sum formulas for the fusion multiplicities.

We also discuss the associated threshold levels, and for  $su(3)$  and  $su(4)$  work out explicitly the minimum threshold level  $t^{\min}$ , and the maximum threshold level  $t^{\max}$ . In the case of  $su(4)$  these are new results. By construction,  $t^{\max}$  is the minimum level for which the (non-vanishing) tensor product multiplicity and the fusion multiplicity coincide. It is therefore of particular interest to know  $t^{\max}$ . Using a refined depth rule applicable to all simple Lie algebras, we find an upper bound on  $t^{\max}$ . Motivated by our results for  $su(4)$  (and the known results for  $su(2)$  and  $su(3)$ ), we conjecture that the upper bound on  $t^{\max}$  is saturated.

## 2 $su(3)$ fusion multiplicities

An  $su(3)$  BZ triangle, describing a particular coupling (to the singlet  $M_0$ ) associated to the triple tensor product  $M_\lambda \otimes M_\mu \otimes M_\nu$ , is a triangular arrangement of 9 non-negative integers:

$$\begin{array}{ccccc}
 & & m_{13} & & \\
 & n_{12} & & l_{23} & \\
 & m_{23} & & m_{12} & \\
 n_{13} & l_{12} & & n_{23} & l_{13}
 \end{array} \tag{7}$$

These integers are related to the Dynkin labels of the three integrable highest weights by

$$\begin{aligned}
 m_{13} + n_{12} &= \lambda_1, & n_{13} + l_{12} &= \mu_1, & l_{13} + m_{12} &= \nu_1, \\
 m_{23} + n_{13} &= \lambda_2, & n_{23} + l_{13} &= \mu_2, & l_{23} + m_{13} &= \nu_2.
 \end{aligned} \tag{8}$$

We call these relations outer constraints. The entries further satisfy the so-called hexagon identities

$$\begin{aligned}
 n_{12} + m_{23} &= n_{23} + m_{12}, \\
 m_{12} + l_{23} &= m_{23} + l_{12}, \\
 l_{12} + n_{23} &= l_{23} + n_{12}
 \end{aligned} \tag{9}$$

of which only two are independent.

In the case of  $su(3)$  there is one basis virtual triangle

$$\mathcal{V} = \begin{array}{ccccc}
 & & 1 & & \\
 & \bar{1} & & \bar{1} & \\
 & \bar{1} & & \bar{1} & \\
 1 & \bar{1} & & \bar{1} & 1
 \end{array} \tag{10}$$

where  $\bar{1} \equiv -1$ . It is easy to work out an initial triangle (a choice of initial triangle valid for all  $su(r+1)$ , may be found in [3]):

$$\mathcal{T}_0 = \begin{array}{ccccc} & & N'_2 & & \\ & n_2 & N_2 & & \\ \lambda_2 & & N'_1 & & \\ 0 & \mu_1 & n_1 & N_1 & \end{array} \quad (11)$$

where

$$\begin{aligned} n_1 &= \lambda^2 + \mu^2 - \nu^1, \quad n_2 = \lambda^1 + \mu^1 - \nu^2, \\ N_1 &= -n_1 + \mu_2, \quad N_2 = n_1 - n_2 + \mu_1, \quad N'_i = \nu_i - N_i, \quad i = 1, 2. \end{aligned} \quad (12)$$

The dual Dynkin labels of the  $su(3)$  weight  $\lambda$  are

$$\lambda^1 = \frac{1}{3}(2\lambda_1 + \lambda_2), \quad \lambda^2 = \frac{1}{3}(\lambda_1 + 2\lambda_2). \quad (13)$$

In general, ordinary and dual Dynkin labels are defined by

$$\lambda = \sum_{i=1}^r \lambda_i \Lambda^i = \sum_{i=1}^r \lambda^i \alpha_i^\vee, \quad (14)$$

where  $\{\Lambda^i\}$  and  $\{\alpha_i^\vee\}$  are the sets of fundamental weights and simple co-roots, respectively.  $r$  is the rank of the Lie algebra. It is  $r = N - 1$  for  $su(N)$ . The set of simple roots is denoted  $\{\alpha_i\}$ .

Any generalised triangle may now be expressed as

$$\mathcal{T} = \mathcal{T}_0 + v\mathcal{V}. \quad (15)$$

Re-imposing that all entries of  $\mathcal{T}$  must be *non-negative* integers, defines a convex polytope (in this case a line segment) characterised by the inequalities

$$0 \leq N'_2 + v, \quad n_2 - v, \quad \lambda_2 - v, \quad v, \quad \mu_1 - v, \quad n_1 - v, \quad N_1 + v, \quad N'_1 - v, \quad N_2 - v. \quad (16)$$

Its discretised volume is the tensor product multiplicity  $T_{\lambda,\mu,\nu}$ . Note the implicit consistency conditions

$$S_i \equiv \lambda^i + \mu^i + \nu^i \in \mathbb{Z}_{\geq}, \quad i = 1, 2, \quad (17)$$

which must be respected to have a non-vanishing multiplicity.

It was shown in Ref. [6] that one can assign the threshold value

$$t = \max\{\lambda_1 + \lambda_2 + l_{13}, \quad \mu_1 + \mu_2 + m_{13}, \quad \nu_1 + \nu_2 + n_{13}\}. \quad (18)$$

to any given  $su(3)$  BZ triangle (7). This means that the fusion multiplicity may be described by supplementing the tensor product conditions (16) by the affine condition

$$k \geq t. \quad (19)$$

Thus, the discretised volume of the convex polytope defined by the inequalities

$$\begin{aligned} 0 \leq & N'_2 + v, \ n_2 - v, \ \lambda_2 - v, \ v, \ \mu_1 - v, \ n_1 - v, \ N_1 + v, \ N'_1 - v, \ N_2 - v, \\ & k - \lambda_1 - \lambda_2 - N_1 - v, \ k - \mu_1 - \mu_2 - N'_2 - v, \ k - \nu_1 - \nu_2 - v, \end{aligned} \quad (20)$$

is the *fusion* multiplicity  $N_{\lambda,\mu,\nu}^{(k)}$ . The volume is easily measured explicitly, and we are left with the sum:

$$\begin{aligned} N_{\lambda,\mu,\nu}^{(k)} &= \sum_{v=v^L}^{v^U} 1, \\ v^L &= \max\{0, -\lambda^1 + \lambda^2 + \mu^1 - \nu^2, \lambda^2 + \mu^1 - \mu^2 - \nu^1\}, \\ v^U &= \min\{\lambda_2, \mu_1, \lambda^2 + \mu^2 - \nu^1, \lambda^1 + \mu^1 - \nu^2, \lambda^2 + \mu^1 - \mu^2 + \nu^1 - \nu^2, \\ &\quad -\lambda^1 + \lambda^2 + \mu^1 - \nu^1 + \nu^2, k - \lambda^1 + \mu^1 - \mu^2 - \nu^1, \\ &\quad k - \lambda^1 + \lambda^2 - \mu^2 - \nu^2, k - \nu^1 - \nu^2\}. \end{aligned} \quad (21)$$

For ease of use, it is expressed entirely in terms of the level  $k$  and the dual and ordinary Dynkin labels. This expression is of course not unique due to the *choice* of initial triangle (11) and the many possible ways of re-writing it. We recall the consistency conditions (17).

## 2.1 Threshold levels

As an application of (21) we may address the question when the fusion multiplicity is greater than any given (non-negative) integer  $M$ . The answer is easily obtained since it corresponds to requiring that  $v^U - v^L \geq M$ , and we find the 27 conditions

$$\begin{aligned} M &\leq \lambda_i, \mu_i, \nu_i, \quad i = 1, 2, 3, \\ M &\leq S_i - (\lambda_1 + \lambda_2), \ S_i - (\mu_1 + \mu_2), \ S_i - (\nu_1 + \nu_2), \\ &\quad S_i - (\lambda_i + \mu_i), \ S_i - (\mu_i + \nu_i), \ S_i - (\nu_i + \lambda_i), \quad i = 1, 2, 3, \\ M &\leq k - (\lambda_1 + \lambda_2), \ k - (\mu_1 + \mu_2), \ k - (\nu_1 + \nu_2), \\ M &\leq k - S_i + \lambda_i, \ k - S_i + \mu_i, \ k - S_i + \nu_i, \quad i = 1, 2, 3. \end{aligned} \quad (22)$$

This is particularly interesting when  $M = 0$ . It also allows us to re-express the fusion multiplicity itself. From (22) it follows immediately that necessary conditions for  $N_{\lambda,\mu,\nu}^{(k)} > 0$  are  $k \geq \lambda_1 + \lambda_2, \mu_1 + \mu_2, \nu_1 + \nu_2, S_i - \min\{\lambda_i, \mu_i, \nu_i\}, i = 1, 2$ , while the fusion multiplicity is equal to the tensor product multiplicity (i.e.,  $N_{\lambda,\mu,\nu}^{(k)}$  is independent of the level  $k$ ) when  $k \geq \min\{S_1, S_2\}$ . Furthermore, according to (18) and the structure of the virtual triangle (10), adding a virtual triangle to a triangle increases the threshold level by one. We conclude that the fusion multiplicity is

$$\begin{aligned} N_{\lambda,\mu,\nu}^{(k)} &= \begin{cases} 0 & \text{if } k < t^{\min} \text{ or } t^{\max} < t^{\min} \\ k - t^{\min} + 1 & \text{if } t^{\min} \leq k \leq t^{\max} \\ T_{\lambda,\mu,\nu} = t^{\max} - t^{\min} + 1 & \text{if } t^{\min} \leq t^{\max} < k \end{cases} \\ t^{\min} &= \max\{\lambda_1 + \lambda_2, \mu_1 + \mu_2, \nu_1 + \nu_2, S_1 - \min\{\lambda_1, \mu_1, \nu_1\}, S_2 - \min\{\lambda_2, \mu_2, \nu_2\}\}, \\ t^{\max} &= \min\{S_1, S_2\}, \end{aligned} \quad (23)$$

still provided (17). The set of threshold levels for the  $T_{\lambda,\mu,\nu}$  distinct couplings is easily read off:

$$\{t^{\min}, t^{\min} + 1, \dots, t^{\max}\} . \quad (24)$$

This confirms the result of [7] where the expression (23) and the threshold level string (24) first appeared.

### 3 $su(4)$ fusion multiplicities

For  $su(4)$  a BZ triangle is defined in terms of 18 non-negative integers:

$$\begin{array}{ccccccc} & & m_{14} & & & & \\ & & n_{12} & l_{34} & & & \\ & m_{24} & & m_{13} & & & \\ n_{13} & l_{23} & n_{23} & l_{24} & & & \\ m_{34} & & m_{23} & & m_{12} & & \\ n_{14} & l_{12} & n_{24} & l_{13} & n_{34} & l_{14} & \end{array} \quad (25)$$

related to the Dynkin labels by

$$\begin{aligned} m_{14} + n_{12} &= \lambda_1 , & n_{14} + l_{12} &= \mu_1 , & l_{14} + m_{12} &= \nu_1 , \\ m_{24} + n_{13} &= \lambda_2 , & n_{24} + l_{13} &= \mu_2 , & l_{24} + m_{13} &= \nu_2 , \\ m_{34} + n_{14} &= \lambda_3 , & n_{34} + l_{14} &= \mu_3 , & l_{34} + m_{14} &= \nu_3 . \end{aligned} \quad (26)$$

The  $su(4)$  BZ triangle contains three hexagons with the associated constraints:

$$\begin{aligned} n_{12} + m_{24} &= m_{13} + n_{23} , & n_{13} + l_{23} &= l_{12} + n_{24} , & l_{24} + n_{23} &= l_{13} + n_{34} , \\ n_{12} + l_{34} &= l_{23} + n_{23} , & n_{13} + m_{34} &= n_{24} + m_{23} , & n_{23} + m_{23} &= m_{12} + n_{34} , \\ m_{24} + l_{23} &= l_{34} + m_{13} , & m_{34} + l_{12} &= l_{23} + m_{23} , & l_{13} + m_{23} &= l_{24} + m_{12} . \end{aligned} \quad (27)$$

Only 6 of these 9 hexagon identities are independent.

In the case of  $su(4)$  the three basis virtual triangles  $\mathcal{V}_1$ ,  $\mathcal{V}_2$  and  $\mathcal{V}_3$  are

$$\begin{aligned} \mathcal{V}_1 &= \begin{array}{cccccc} & & 1 & & & \\ & \bar{1} & & \bar{1} & & \\ & \bar{1} & & \bar{1} & & \\ 1 & \bar{1} & \bar{1} & 1 & & \\ 0 & & 1 & & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} & \mathcal{V}_2 &= \begin{array}{cccccc} & & 0 & & & \\ & 0 & & 0 & & \\ & 1 & & 0 & & \\ \bar{1} & \bar{1} & 1 & 0 & & \\ \bar{1} & & \bar{1} & & 0 & \\ 1 & \bar{1} & \bar{1} & 1 & 0 & 0 \end{array} \\ \mathcal{V}_3 &= \begin{array}{cccccc} & & 0 & & & \\ & 0 & & 0 & & \\ & 0 & & 1 & & \\ 0 & 1 & \bar{1} & \bar{1} & & \\ 0 & & \bar{1} & & \bar{1} & \\ 0 & 0 & 1 & \bar{1} & \bar{1} & 1 \end{array} \end{aligned} \quad (28)$$

We make the following choice of initial triangle [3]:

$$\mathcal{T}_0 = \begin{array}{ccccccc} & & & & N'_3 & & \\ & & & & n_3 & N_3 & \\ & & \lambda_2 & & & N'_2 & \\ & 0 & \mu_1 & n_2 & & N_2 & \\ & \lambda_3 & & \lambda_3 & & N'_1 & \\ 0 & \mu_1 & 0 & \mu_2 & n_1 & N_1 & \end{array} \quad (29)$$

where

$$\begin{aligned} n_1 &= \lambda^3 + \mu^3 - \nu^1, \quad n_2 = \lambda^2 + \mu^2 - \nu^2, \quad n_3 = \lambda^1 + \mu^1 - \nu^3, \\ N_1 &= -n_1 + \mu_3, \quad N_2 = n_1 - n_2 + \mu_2, \quad N_3 = n_2 - n_3 + \mu_1, \quad N'_i = \nu_i - N_i, \quad i = 1, 2, 3. \end{aligned} \quad (30)$$

The dual Dynkin labels (14) of the  $su(4)$  weight  $\lambda$  are

$$\lambda^1 = \frac{1}{4}(3\lambda_1 + 2\lambda_2 + \lambda_3), \quad \lambda^2 = \frac{1}{4}(2\lambda_1 + 4\lambda_2 + 2\lambda_3), \quad \lambda^3 = \frac{1}{4}(\lambda_1 + 2\lambda_2 + 3\lambda_3). \quad (31)$$

Now, any generalised triangle may be written

$$\mathcal{T} = \mathcal{T}_0 + \sum_{l=1}^3 v_l \mathcal{V}_l, \quad (32)$$

and the tensor product multiplicity  $T_{\lambda, \mu, \nu}$  is the discretised volume of the (in general) three-dimensional convex polytope

$$\begin{aligned} 0 \leq & \quad v_2, \quad \mu_1 - v_2, \quad \lambda_3 - v_2, \quad -v_2 + v_3, \quad v_1 - v_2, \quad \mu_2 + v_2 - v_3, \quad \lambda_2 - v_1 + v_2, \\ & \lambda_3 + v_1 - v_2 - v_3, \quad \mu_1 - v_1 - v_2 + v_3, \quad n_1 - v_3, \quad n_2 - v_1 + v_2 - v_3, \quad n_3 - v_1, \\ & N_1 + v_3, \quad N'_1 - v_3, \quad N_2 + v_1 - v_3, \quad N'_2 - v_1 + v_3, \quad N_3 - v_1, \quad N'_3 + v_1. \end{aligned} \quad (33)$$

We have the consistency conditions

$$S_i \equiv \lambda^i + \mu^i + \nu^i \in \mathbb{Z}_{\geq}, \quad i = 1, 2, 3. \quad (34)$$

It was shown in Ref. [8] (and confirmed in [9]) that one can assign the threshold level

$$\begin{aligned} t = & \max\{\lambda_1 + \lambda_2 + \lambda_3 + l_{14}, \quad \mu_1 + \mu_2 + \mu_3 + m_{14}, \quad \nu_1 + \nu_2 + \nu_3 + n_{14}, \\ & \lambda_1 + \lambda_2 + l_{14} + l_{24} + n_{14}, \quad \lambda_2 + \lambda_3 + l_{14} + l_{13} + m_{14}, \\ & \mu_1 + \mu_2 + m_{14} + m_{24} + l_{14}, \quad \mu_2 + \mu_3 + m_{14} + m_{13} + n_{14}, \\ & \nu_1 + \nu_2 + n_{14} + n_{24} + m_{14}, \quad \nu_2 + \nu_3 + n_{14} + n_{13} + l_{14}, \\ & l_{14} + m_{14} + n_{14} + [\frac{1}{2}(\lambda_2 + \mu_2 + \nu_2 + l_{23} + m_{23} + n_{23} + 1)]\} \end{aligned} \quad (35)$$

to any given  $su(4)$  BZ triangle (25).  $[x]$  denotes the integer value of  $x$ , i.e., the greatest integer less than or equal to  $x$ . This means that the fusion multiplicity may be described by supplementing the tensor product conditions (33) by the affine condition (19), with (35) as the threshold level  $t$ .

After a simple re-writing, the condition  $k \geq t$  on the last line of (35) becomes

$$k \geq \left\lceil \frac{1}{2}(\lambda^2 + \mu^2 + \nu^2 + l_{14} + m_{14} + n_{14} + 1) \right\rceil. \quad (36)$$

According to (32), it involves the integer value of a possibly half-integer number depending on the parameters  $v_l$  defining the polytope. Thus, the polytope is in general not convex, and measuring its volume is not straightforward. However, we observe that for integer  $k$  and  $B$ , the condition  $k \geq [B/2]$  is equivalent to  $k \geq (B-1)/2$ . The condition (36) may therefore be simplified as

$$v_1 + v_2 + v_3 \leq 2k - \lambda^1 + \lambda^3 + \mu^1 - \mu^3 - \nu^1 - \nu^2 - \nu^3. \quad (37)$$

In conclusion, the discretised volume of the *convex* polytope defined by  $v_1$ ,  $v_2$  and  $v_3$  subject to the inequalities

$$\begin{aligned} 0 \leq & v_2, \mu_1 - v_2, \lambda_3 - v_2, -v_2 + v_3, v_1 - v_2, \mu_2 + v_2 - v_3, \lambda_2 - v_1 + v_2, \\ & \lambda_3 + v_1 - v_2 - v_3, \mu_1 - v_1 - v_2 + v_3, n_1 - v_3, n_2 - v_1 + v_2 - v_3, n_3 - v_1, \\ & N_1 + v_3, N'_1 - v_3, N_2 + v_1 - v_3, N'_2 - v_1 + v_3, N_3 - v_1, N'_3 + v_1, \\ & k - \lambda_1 - \lambda_2 - \lambda_3 - N_1 - v_3, k - \mu_1 - \mu_2 - \mu_3 - N'_3 - v_1, k - \nu_1 - \nu_2 - \nu_3 - v_2, \\ & k - \lambda_1 - \lambda_2 - N_1 - N_2 - v_1 - v_2, k - \mu_2 - \mu_3 - N'_2 - N'_3 - v_2 - v_3, \\ & k + \lambda_3 - \nu_1 - \nu_2 - \nu_3 - v_2 - v_3, k + \mu_1 - \nu_1 - \nu_2 - \nu_3 - v_1 - v_2, \\ & k - \nu_1 - \nu_2 - N'_3 - v_1 - v_3, k - \nu_2 - \nu_3 - N_1 - v_1 - v_3, \\ & 2k - \lambda^1 + \lambda^3 + \mu^1 - \mu^3 - \nu^1 - \nu^2 - \nu^3 - v_1 - v_2 - v_3, \end{aligned} \quad (38)$$

is the fusion multiplicity  $N_{\lambda, \mu, \nu}^{(k)}$ . The volume is easily measured explicitly, and we are left with the multiple sum:

$$\begin{aligned} N_{\lambda, \mu, \nu}^{(k)} &= \sum_{v_2=v_2^L}^{v_2^U} \sum_{v_1=v_1^L}^{v_1^U} \sum_{v_3=v_3^L}^{v_3^U} 1, \\ v_3^L &= \max\{v_2, -\mu_1 + v_1 + v_2, \lambda^3 + \mu^2 - \mu^3 - \nu^1, -\lambda^2 + \lambda^3 - \mu^1 + \mu^2 - \nu^2 + \nu^3 + v_1\}, \\ v_3^U &= \min\{\mu_2 + v_2, \lambda_3 + v_1 - v_2, \lambda^3 + \mu^3 - \nu^1, \lambda^2 + \mu^2 - \nu^2 - v_1 + v_2, \\ & \quad \lambda^3 + \mu^2 - \mu^3 + \nu^1 - \nu^2, -\lambda^2 + \lambda^3 - \mu^1 + \mu^2 - \nu^1 + \nu^2 + v_1, \\ & \quad k - \lambda^1 + \lambda^3 + \mu^1 - \mu^3 - \nu^2 - v_2, k + \lambda_3 - \nu_1 - \nu_2 - \nu_3 - v_2, \\ & \quad k - \lambda^1 + \lambda^2 + \mu^1 - \nu^1 - \nu^2 - v_1, k + \lambda^3 + \mu^2 - \mu^3 - \nu^2 - \nu^3 - v_1, \\ & \quad k - \lambda^1 + \mu^2 - \mu^3 - \nu^1, 2k - \lambda^1 + \lambda^3 + \mu^1 - \mu^3 - \nu^1 - \nu^2 - \nu^3 - v_1 - v_2\}, \\ v_1^L &= \max\{v_2, -\lambda^1 + \lambda^2 + \mu^1 - \nu^3\}, \\ v_1^U &= \min\{\lambda_2 + v_2, \lambda^1 + \mu^1 - \nu^3, -\lambda^1 + \lambda^2 + \mu^1 - \nu^2 + \nu^3, k - \lambda^1 + \lambda^2 - \mu^3 - \nu^3, \\ & \quad k - \lambda^1 + \lambda^3 + \mu^1 - \mu^3 - \nu^2 - v_2, k + \mu_1 - \nu_1 - \nu_2 - \nu_3 - v_2\}, \\ v_2^L &= 0, \\ v_2^U &= \min\{\mu_1, \lambda_3, k - \nu_1 - \nu_2 - \nu_3\}. \end{aligned} \quad (39)$$

For ease of use, it is expressed entirely in terms of the level  $k$  and the dual and ordinary Dynkin labels. This is the most explicit result for affine  $su(4)$  fusion multiplicities that we know of. The



choice of order of summation in (39) is immaterial, and we recall the consistency conditions (34) which are required for a non-vanishing fusion multiplicity. They also ensure that all bounds are integer.

### 3.1 Threshold levels

An obvious property of the convex polytope characterising the  $su(4)$  fusion multiplicity is that it is connected. Furthermore, we have seen that the fusion polytope (38) corresponds to “slicing out” a convex polytope embedded in the tensor product polytope (33). The slicing procedure involved then shows us that the support of the threshold multiplicities is also connected, i.e., for non-vanishing tensor product multiplicity we have:

$$\text{For } T_{\lambda,\mu,\nu} > 0 : \quad n_{\lambda,\mu,\nu}^{(t)} > 0 \iff t^{\min} \leq t \leq t^{\max} \quad (40)$$

Even though this is a very natural result, we believe that our convex polytope description has provided the most convincing evidence hitherto. It is still not a rigorous proof since a rigorous proof of (35) has yet to be found.

The argument leading to (40) relies solely on the connectedness of the tensor product polytope, and the assignment of a threshold level to a true BZ triangle as in (35):  $t$  is the *maximum* of a set of expressions in the entries. In Ref. [3] we have shown that the tensor product multiplicities for all  $su(r+1)$  may be characterised by convex polytopes, i.e., in particular connected polytopes. Since we believe that it is possible for all  $su(r+1)$  to assign a threshold level to any true BZ triangle as a maximum of a set of expressions in the entries, we conjecture that (40) is valid for all  $su(r+1)$ .

The slicing procedure invites us to make a further conjecture concerning the threshold multiplicities for a non-vanishing tensor product multiplicity:

$$\text{For } T_{\lambda,\mu,\nu} > 0, \exists t_0 \in \mathbb{Z}_{\geq} : \quad \begin{cases} n_{\lambda,\mu,\nu}^{(t-1)} \leq n_{\lambda,\mu,\nu}^{(t)}, & t \leq t_0 \\ n_{\lambda,\mu,\nu}^{(t)} \geq n_{\lambda,\mu,\nu}^{(t+1)}, & t > t_0 \end{cases} \quad (41)$$

where  $n_{\lambda,\mu,\nu}^{(t < 0)} \equiv 0$ . This conjecture indicates that the threshold multiplicity  $n_{\lambda,\mu,\nu}^{(t)}$  as a function of  $t$  has exactly one local maximum. It is a refinement of (40) since it implies (40). It is made plausible by the observation that the slicing procedure cuts off pieces of the polytope “from above”: all the affine conditions in (38) correspond to planes with oriented normal vectors  $(-1, 0, 0)$ ,  $(0, -1, 0)$ ,  $(0, 0, -1)$ ,  $(-1, -1, 0)$ ,  $(-1, 0, -1)$ ,  $(0, -1, -1)$  or  $(-1, -1, -1)$ . Since the polytope is *convex*, the slicing procedure for decreasing  $k$  will in general cut off bigger and bigger pieces until a certain point, after which the pieces will become smaller and smaller. A maximum sized piece is cut off when  $k$  decreases from  $t_0$  to  $t_0 - 1$ . Note that  $t_0$  may be any integer in the interval  $[t^{\min}, t^{\max}]$ . In general there will be a (non-vanishing and connected) sub-interval  $[t_0^{\min}, t_0^{\max}]$ , where any integer in it may play the role of  $t_0$ . In most of the cases we have analysed explicitly,  $t_0^{\min} = t_0^{\max}$ , though that is not always true. Examples are provided below.

We conjecture that (41) is valid for all  $su(r+1)$ . So far, this conjecture has passed all the non-trivial tests we have made, though a proof is still lacking.

The minimum threshold level  $t^{\min}$  may be computed by determining necessary and sufficient conditions for the fusion multiplicity  $N_{\lambda,\mu,\nu}^{(k)}$  to be non-vanishing. Determining those, involves a straightforward, though cumbersome, analysis of the convex polytope (38) or equivalently of the multiple sum formula (39) – see also [3]. The resulting conditions may be expressed as a set of inequalities in the (ordinary and dual) Dynkin labels and the level  $k$ . We choose to present the result in terms of the Weyl group, since we believe that a similar and universal characterisation exists, valid for all simple Lie algebras.

The Weyl group  $W$  is generated by three simple reflections:

$$s_i \lambda = \lambda - \lambda_i \alpha_i, \quad i = 1, 2, 3. \quad (42)$$

This extends readily to all simple Lie algebras.

Now, we find that  $N_{\lambda,\mu,\nu}^{(k)}$  is non-vanishing provided

$$\begin{aligned} 0 &\leq \lambda_i, \mu_i, \nu_i, \quad i = 1, 2, 3, \\ 0 &\leq \Lambda^1 \cdot (u\lambda + v\mu + w\nu), \quad u, v, w \in \{I, s_1, s_1 s_2, s_1 s_2 s_3\}, \\ &\quad l(u) + l(v) + l(w) = 3, \\ 0 &\leq \Lambda^2 \cdot (u\lambda + v\mu + w\nu), \quad u, v, w \in \{I, s_2, s_2 s_1, s_2 s_1 s_3, s_2 s_1 s_3 s_2\}, \\ &\quad l(u) + l(v) + l(w) = 4, \\ 0 &\leq \Lambda^2 \cdot (u\lambda + v\mu + w\nu), \quad u, v, w \in \{I, s_2, s_2 s_3, s_2 s_3 s_1, s_2 s_3 s_1 s_2\}, \\ &\quad l(u) + l(v) + l(w) = 4, \\ 0 &\leq \Lambda^3 \cdot (u\lambda + v\mu + w\nu), \quad u, v, w \in \{I, s_3, s_3 s_2, s_3 s_2 s_1\}, \\ &\quad l(u) + l(v) + l(w) = 3 \end{aligned} \quad (43)$$

and

$$\begin{aligned} 0 &\leq \lambda_0, \mu_0, \nu_0, \\ 2k &\geq S_2, \\ k &\geq \Lambda^1 \cdot (u\lambda + v\mu + w\nu), \quad u, v, w \in \{I, s_1, s_1 s_2\}, \quad l(u) + l(v) + l(w) = 2, \\ k &\geq \Lambda^2 \cdot (u\lambda + v\mu + w\nu), \\ &\quad (u, v, w) \in \{p(I, s_2, s_2 s_1 s_3), p(I, s_2 s_1, s_2 s_3), p(s_2, s_2, s_2 s_1), p(s_2, s_2, s_2 s_3) \mid p \in \mathcal{S}_3\}, \\ k &\geq \Lambda^3 \cdot (u\lambda + v\mu + w\nu), \quad u, v, w \in \{I, s_3, s_3 s_2\}, \quad l(u) + l(v) + l(w) = 2. \end{aligned} \quad (44)$$

$I$  is the identity (i.e.,  $I\lambda = \lambda$ ),  $\mathcal{S}_3$  is the permutation group of three elements, while  $l(u)$  denotes the length of the Weyl group element  $u$  (with  $l(I) = 0$ ). Note that the two  $\Lambda^2$ -conditions in (43) contain identical conditions on the weights. They are included to keep the expression compact and symmetric. (43) contains altogether 50 independent constraints (in [3] they are expressed as bounds on the weight  $\nu$ ), while (44) contains 34 independent constraints.

To be clear, we stress that it is the Weyl orbits of the fundamental weights which are important here. The Weyl group elements used above, and their particular expressions as products of simple reflections  $s_j$  (words), are not unique.

At least some of the inequalities (43) and (44) are quite easily understood. For example,  $N_{\lambda,\mu,\nu}^{(k)} \neq 0$  implies that  $\nu^+ - \lambda \in P(\mu)$ , the set of weights (of non-vanishing multiplicity) of the

module  $M_\mu$ . The boundaries of the weight diagram of  $M_\mu$  are easily described:

$$\tilde{\mu} \in P(\mu) \Rightarrow \mu \cdot \Lambda^j - \tilde{\mu} \cdot (w\Lambda^j) \geq 0, \quad j = 1, \dots, r, \quad w \in W. \quad (45)$$

Using  $\nu^+ = -w_0\nu$ , we find

$$0 \leq \Lambda^j \cdot (w\lambda + \mu + ww_0\nu), \quad (46)$$

for any  $w \in W$ . Other necessary inequalities can be written by permuting  $\lambda, \mu, \nu$  in (46).

These are far from sufficient as conditions for non-vanishing tensor product multiplicities, however. In addition, they do not include any level-dependent constraints for fusion, like (44). But fusion multiplicities may be found from certain formulas for tensor product multiplicities by replacing the Weyl group  $W$  by the projection of the affine Weyl group. We suspect that the  $k$ -dependent inequalities may be explained this way.

There is a long history of the problem of finding conditions on highest weights such that the corresponding tensor product multiplicity is non-zero. For a review, see [10]. Recent advances include work on the tensor products for  $GL_n(\mathbb{C})$ . Necessary and sufficient “non-vanishing conditions” turn out to have a relatively simple description in terms of Schubert calculus. The corresponding fusion problem has a similar solution, involving quantum Schubert calculus [11]. The connection between fusion and quantum cohomology goes back to Witten [12].

As a simple application of (44),  $t^{\min}$  is found to be

$$\begin{aligned} t^{\min} = \max\{ & \lambda^1 + \lambda^3, \mu^1 + \mu^3, \nu^1 + \nu^3, [\frac{1}{2}(\lambda^2 + \mu^2 + \nu^2 + 1)], \\ & \lambda^1 + \mu^1 + \nu_3 - \nu^3, \lambda^1 + \mu_3 - \mu^3 + \nu^1, \lambda_3 - \lambda^3 + \mu^1 + \nu^1, \\ & \lambda^3 + \mu^3 + \nu_1 - \nu^1, \lambda^3 + \mu_1 - \mu^1 + \nu^3, \lambda_1 - \lambda^1 + \mu^3 + \nu^3, \\ & \lambda^1 - \mu_1 + \mu^1 - \nu_1 + \nu^1, -\lambda_1 + \lambda^1 + \mu^1 - \nu_1 + \nu^1, -\lambda_1 + \lambda^1 - \mu_1 + \mu^1 + \nu^1, \\ & \lambda^3 + \mu^3 - \mu_3 + \nu^3 - \nu_3, \lambda^3 - \lambda_3 + \mu^3 + \nu^3 - \nu_3, \lambda^3 - \lambda_3 + \mu^3 - \mu_3 + \nu^3, \\ & \lambda^2 - \mu^1 + \mu^3 + \nu^1 - \nu^3, -\lambda^1 + \lambda^3 + \mu^2 + \nu^1 - \nu^3, -\lambda^1 + \lambda^3 + \mu^1 - \mu^3 + \nu^2, \\ & \lambda^2 + \mu^1 - \mu^3 - \nu^1 + \nu^3, \lambda^1 - \lambda^3 + \mu^2 - \nu^1 + \nu^3, \lambda^1 - \lambda^3 - \mu^1 + \mu^3 + \nu^2, \\ & \lambda^2 - \lambda_2 - \mu^2 + \mu_2 + \nu^2, -\lambda^2 + \lambda_2 + \mu^2 - \mu_2 + \nu^2, \lambda^2 - \lambda_2 + \mu^2 - \nu^2 + \nu_2, \\ & -\lambda^2 + \lambda_2 + \mu^2 + \nu^2 - \nu_2, \lambda^2 + \mu^2 - \mu_2 - \nu^2 + \nu_2, \lambda^2 - \mu^2 + \mu_2 + \nu^2 - \nu_2, \\ & \lambda^2 - \lambda_2 + \mu^2 - \mu_2 - \nu^1 + \nu^3, \lambda^2 - \lambda_2 + \mu^2 - \mu_2 + \nu^1 - \nu^3, \\ & \lambda^2 - \lambda_2 - \mu^1 + \mu^3 + \nu^2 - \nu_2, \lambda^2 - \lambda_2 + \mu^1 - \mu^3 + \nu^2 - \nu_2, \\ & -\lambda^1 + \lambda^3 + \mu^2 - \mu_2 + \nu^2 - \nu_2, \lambda^1 - \lambda^3 + \mu^2 - \mu_2 + \nu^2 - \nu_2\}. \end{aligned} \quad (47)$$

We find that the maximum threshold level  $t^{\max}$  is

$$t^{\max} = \min\{S_1, S_2 - \max\{\lambda_2, \mu_2, \nu_2\}, S_3\} \quad (48)$$

This formula first appeared in [13], though without proof. According to the general discussion below on upper bounds on  $t^{\max}$ , (48) states that the relevant bound (62) for  $su(4)$  is saturated. To prove that explicitly, we show that for non-vanishing tensor product multiplicity  $T_{\lambda, \mu, \nu}$ , there is at least one integer point in the associated convex polytope that corresponds to a (true) BZ triangle of threshold level  $t$  (35) equal to the conjectured value (48).

Following the discussion above we should consider a point on the surface of the polytope. The surface is characterised by one or several of the BZ entries (25) being zero. When at least two entries vanish simultaneously, we are at an intersection of faces. Since the vanishing of a corner point ( $m_{14} = 0$ ,  $n_{14} = 0$  or  $l_{14} = 0$ ) corresponds to a minimum value of  $v_1$ ,  $v_2$  or  $v_3$ , symmetry then allows us to consider three initial conditions only:  $n_{12} = 0$ ,  $m_{24} = 0$  or  $l_{23} = 0$ . To indicate how the proof goes, let us assume that  $n_{12} = 0$ . We may then add the sum  $\mathcal{V}_2 + \mathcal{V}_3$  to the triangle (leaving  $n_{12} = 0$  unchanged) until one of the entries  $n_{13}$ ,  $m_{34}$ ,  $l_{12}$ ,  $n_{34}$ ,  $m_{12}$ ,  $l_{24}$  or  $m_{23}$  also vanish, or  $m_{23} = 1$ . Depending on the situation one may then add additional numbers of  $\mathcal{V}_2$  or  $\mathcal{V}_3$  to obtain yet another vanishing entry. With sufficiently many vanishing entries, the threshold level (35) always coincides with (48). A priori, there are many possible intersections to analyze, but a straightforward and systematic approach makes the analysis tractable. In that way we have shown that the maximum threshold level is given by the simple expression (48).

### 3.2 Examples

The threshold level is in general not linear in the BZ triangles:

$$t(\mathcal{T}_1 + \mathcal{T}_2) \leq t(\mathcal{T}_1) + t(\mathcal{T}_2) \quad (49)$$

Here  $\mathcal{T}_1$  and  $\mathcal{T}_2$  indicate true BZ triangles. Of particular interest is the behavior under addition of virtual triangles. As is easily seen, the inequality (49) is in general not saturated even in those cases:

$$\begin{aligned} t(\mathcal{T} + \mathcal{V}_i) &= t(\mathcal{T}) + c, & c \in \{0, 1\}, \\ t(\mathcal{T} + \mathcal{W} - \mathcal{V}_i) &= t(\mathcal{T}) + c, & c \in \{0, 1, 2\}, \\ t(\mathcal{T} + \mathcal{W}) &= t(\mathcal{T}) + c, & c \in \{1, 2\}, \end{aligned} \quad (50)$$

whereas formally we may assign the threshold levels

$$t(\mathcal{V}_i) = 1, \quad t(\mathcal{W} - \mathcal{V}_i) = t(\mathcal{W}) = 2. \quad (51)$$

Here we have introduced the linear combination

$$\mathcal{W} \equiv \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 = \begin{array}{ccccc} & & 1 & & \\ & & \bar{1} & & \bar{1} \\ & 0 & & 0 & \\ & 0 & \bar{1} & \bar{1} & 0 \\ \bar{1} & & \bar{1} & & \bar{1} \\ 1 & \bar{1} & 0 & 0 & \bar{1} & 1 \end{array} \quad (52)$$

Note that for  $su(3)$  we have  $t(\mathcal{T} + \mathcal{V}) = t(\mathcal{T}) + 1$  for all triangles. In the following we will discuss two examples where a similarly simple situation occurs for  $su(4)$ .

Our first example is defined by  $\lambda_2 = \mu_2 = 0$ , with the remaining seven non-negative Dynkin labels restricted a priori only by (34). In this case the convex polytope is one-dimensional and generated by  $\mathcal{W}$  (52). It then follows from (40) that  $t(\mathcal{T} + \mathcal{W}) = t(\mathcal{T}) + 1$ , and we conclude that we are in a situation similar to the general  $su(3)$  case – see (23) and (24):

$$N_{\lambda, \mu, \nu}^{(k < t^{\max})} = \sum_{j=t^{\min}}^k 1, \quad N_{\lambda, \mu, \nu}^{(k \geq t^{\max})} = T_{\lambda, \mu, \nu} = \sum_{j=t^{\min}}^{t^{\max}} 1. \quad (53)$$

The set of associated threshold levels is

$$\{t^{\min}, t^{\min} + 1, \dots, t^{\max}\} \quad (54)$$

In particular, all non-vanishing threshold multiplicities are 1, and  $t_0^{\min} = t^{\min}$  and  $t_0^{\max} = t^{\max}$ . Furthermore, it is easily shown that for  $\lambda_2 = \mu_2 = 0$  the maximum threshold level (48) is

$$t^{\max} = S_2 - \nu_2 . \quad (55)$$

It follows that the tensor product multiplicity may be expressed as the minimum of a simple set of linear combinations of Dynkin labels:

$$\begin{aligned} T_{\lambda, \mu, \nu} = 1 + \min\{ & \lambda_1, \lambda_3, \mu_1, \mu_3, \nu_1, \nu_3, \lambda_1 + \lambda_3 - \nu_2, \mu_1 + \mu_3 - \nu_2, \\ & \lambda_1 + \mu_3 - \nu_2, \lambda_3 + \mu_1 - \nu_2, \lambda^2 + \mu^2 - \nu^2, [\frac{1}{2}S_2] - \nu_2 \\ & \lambda^2 - \mu^2 + \nu^2 - \nu_2, -\lambda^2 + \mu^2 + \nu^2 - \nu_2 \\ & \lambda^1 - \mu^1 + \mu^2 + \nu^1 - \nu_2, -\lambda^1 + \lambda^2 + \mu^1 + \nu^1 - \nu_2, \\ & \lambda^3 + \mu^2 - \mu^3 + \nu^3 - \nu_2, \lambda^2 - \lambda^3 + \mu^3 + \nu^3 - \nu_2, \\ & -\lambda^1 + \lambda^2 - \mu^1 + \mu^2 + \nu^1, \lambda^2 - \lambda^3 + \mu^2 - \mu^3 + \nu^3, \\ & \lambda^1 - \mu^1 + \mu^2 - \nu^2 + \nu^3, -\lambda^1 + \lambda^2 + \mu^1 - \nu^2 + \nu^3, \\ & \lambda^3 + \mu^2 - \mu^3 + \nu^1 - \nu^2, \lambda^2 - \lambda^3 + \mu^3 + \nu^1 - \nu^2\} , \end{aligned} \quad (56)$$

using (47). It is understood that  $T_{\lambda, \mu, \nu}$  vanishes if the expression (56) is negative.

Our second example is defined by  $\lambda_3 = \mu_3 = 0$ , with the remaining seven non-negative Dynkin labels restricted a priori only by (34). The convex polytope is one-dimensional and generated by  $\mathcal{V}_1$  of (28). In the general expression (47) for  $t^{\min}$ , one of the contributors is  $\lambda^3 - \lambda_3 + \mu^3 - \mu_3 + \nu^3$  which in this case reduces to  $S_3$ . We may therefore conclude that

$$t^{\min} = t^{\max} = S_3 \quad (57)$$

It then follows that  $t(\mathcal{T} + \mathcal{V}_1) = t(\mathcal{T})$ , and there is only one non-vanishing threshold multiplicity:  $n_{\lambda, \mu, \nu}^{(S_3)} = T_{\lambda, \mu, \nu}$  and  $t_0^{\min} = t_0^{\max} = S_3$ . A straightforward analysis of the single-sum expressing the tensor product multiplicity yields

$$\begin{aligned} T_{\lambda, \mu, \nu} = 1 + \min\{ & \lambda_1, \mu_1, \nu_1, \nu_2, \nu_3, -\lambda^1 + \lambda^2 - \mu^3 + \nu^1, -\lambda^3 - \mu^1 + \mu^2 + \nu^1, \\ & -\lambda^1 + \mu^3 + \nu^3, \lambda^3 - \mu^1 + \nu^3, \lambda^1 - \mu^3 - \nu^1 + \nu^2, -\lambda^3 + \mu^1 - \nu^1 + \nu^2, \\ & \lambda^1 - \lambda^3 + \mu^1 - \mu^3 + \nu^1 - \nu^3, \lambda^1 - \lambda^3 + \mu^1 - \mu^3 - \nu^1 + \nu^3, \\ & \lambda^1 - \lambda^2 + \mu^3 + \nu^2 - \nu^3, \lambda^3 + \mu^1 - \mu^2 + \nu^2 - \nu^3, -\lambda^1 + \lambda^3 - \mu^1 + \mu^3 + \nu^2, \\ & \lambda^1 - \lambda^3 - \mu^1 + \mu^3 + \nu^1 - \nu^2 + \nu^3, -\lambda^1 + \lambda^3 + \mu^1 - \mu^3 + \nu^1 - \nu^2 + \nu^3\} , \end{aligned} \quad (58)$$

provided  $\lambda^3 + \mu^3 - \nu^1 \geq 0$  and the expression (58) is non-negative. If one of these conditions fails to be satisfied,  $T_{\lambda, \mu, \nu}$  vanishes by construction.

Furthermore, we can write a very simple but necessary condition on the tensor product multiplicity. Assuming the two conditions  $\lambda^3 + \mu^3 - \nu^1 \geq 0$  and  $T_{\lambda, \mu, \nu} \geq 0$  are satisfied, (58) easily yields

$$T_{\lambda, \mu, \nu} \leq 1 + \min\{\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3\} . \quad (59)$$

The two examples above illustrate two extreme situations: a “horizontal” distribution of threshold levels, and a “vertical” distribution. In the first example we have a threshold level string with associated multiplicities all being one – a purely horizontal distribution. In the second example we have exactly one non-vanishing threshold multiplicity – a purely vertical distribution. A generic situation will have a much more complicated distribution, which we believe must respect (40) and (41).

It is trivial to devise an alternative example of a purely vertical distribution, namely  $\lambda_1 = \mu_1 = 0$ . Analogues to the results above in the case  $\lambda_3 = \mu_3 = 0$ , are obtained by conjugation:  $(\lambda_1, \lambda_2, \lambda_3)^+ = (\lambda_3, \lambda_2, \lambda_1)$ , etc. Simple permutations of the three weights  $\lambda$ ,  $\mu$  and  $\nu$  also provide new examples.

## 4 Refined depth rule and threshold levels

A refinement of the Gepner-Witten depth rule

$$N_{\lambda, \mu, \nu}^{(k)} = \dim\{v \in M_{\mu, \nu^+ - \lambda} \mid f_i^{\nu_i + 1} v = 0, i = 1, \dots, r; e_\theta^{k - \nu \cdot \theta + 1} v = 0\} \quad (60)$$

was conjectured in [6, 14] (see also [15]). Here  $M_{\mu, \nu^+ - \lambda}$  is the subspace of  $M_\mu$  of weight  $\nu^+ - \lambda$ ,  $f_i$  is the lowering operator associated to the simple root  $\alpha_i$ , and  $e_\theta$  is the raising operator associated to  $\theta$ , the highest root of the simple Lie algebra (of rank  $r$ ) being considered. When the level  $k$  is large, the constraint  $e_\theta^{k - \nu \cdot \theta + 1} v = 0$  is automatically satisfied, and a well-known formula for the tensor product coefficients is recovered. This agrees with (4).

In this section  $u$ ,  $v$  and  $w$  denote vectors – not Weyl group elements as they did in the previous section.

We now derive an upper bound on the maximum threshold level associated to  $N_{\lambda, \mu, \nu}^{(k)} \neq 0$ . The depth  $d_v \in \mathbb{Z}_{\geq}$  of  $v$  is the minimum possible power of  $e_\theta$  such that  $e_\theta^{d_v + 1} v = 0$ . (60) tells us that if a vector  $v \in M_{\mu, \nu^+ - \lambda}$  associated to a fusion coupling has depth  $d_v$ , then its threshold level is  $t = d_v + \nu \cdot \theta$ .

An obvious upper bound on  $d_v$  is the maximum number of times  $\theta$  can be added to the weight  $\nu^+ - \lambda$  of  $v$  to obtain a “higher weight” still having non-zero multiplicity in  $M_\mu$ . For the upper bound, the higher weight  $\nu^+ - \lambda + d_{v, \max} \theta$  should be a weight on the boundary of the weight diagram. Because  $\theta$  is the highest root, the boundary weight will have  $\Lambda^j \cdot (\nu^+ - \lambda + d_{v, \max} \theta) = \Lambda^j \cdot \mu$ , for some  $j \in \{1, \dots, r\}$ . We write  $\theta = \sum_{j=1}^r a^{\vee j} \alpha_j^\vee$ , where  $a^{\vee j}$  and  $\alpha_j^\vee$  denote the  $j$ -th co-mark and simple co-root, respectively. Then we obtain  $d_{v, \max} = \min\{(\lambda^j + \mu^j - \nu^{j+})/a^{\vee j} \mid j = 1, \dots, r\}$ , where  $\nu^{j+} \equiv \Lambda^j \cdot \nu^+$ . Finally, this gives

$$t^{\max} \leq \min\{(\lambda^j + \mu^j - \nu^{j+})/a^{\vee j} + \nu \cdot \theta \mid j = 1, \dots, r\}. \quad (61)$$

Note that all co-marks  $a_j^\vee$  are non-vanishing.

This bound can be improved by noticing that (60) treats the weights  $\{\lambda, \mu, \nu\}$  asymmetrically. So permuting those weights will result in other upper bounds on  $t^{\max}$ . For  $p \in \mathcal{S}_3$ , let  $p\{\lambda, \mu, \nu\} \equiv \{p\lambda, p\mu, p\nu\}$ . Then we must have

$$t^{\max} \leq \min\left\{\left((p\lambda)^j + (p\mu)^j - (p\nu)^{j+}\right)/a^{\vee j} + (p\nu) \cdot \theta \mid j \in \{1, \dots, r\}, p \in \mathcal{S}_3\right\}. \quad (62)$$

Finally, we conjecture that this bound is saturated for all (untwisted) affine Lie algebras, i.e., (62) is an equality. As shown in previous sections, that is true for  $su(3)$  and  $su(4)$ , and it is easily seen to be true for  $su(2)$ .

Also interesting is a similar argument based on the symmetric form of the depth rule. Suppose  $u \in M_{\lambda, \bar{\lambda}}$ ,  $v \in M_{\mu, \bar{\mu}}$ ,  $w \in M_{\nu, \bar{\nu}}$ , and that the Clebsch-Gordan coefficient  $C_{u,v,w} \neq 0$  for the coupling  $M_{\lambda} \otimes M_{\mu} \otimes M_{\nu} \supset M_0$  of interest. Then  $\bar{\lambda} + \bar{\mu} + \bar{\nu} = 0$ , and the symmetric depth rule says that

$$(e_{\theta})^a u \otimes (e_{\theta})^b v \otimes (e_{\theta})^c w = 0, \text{ for all } a + b + c > k. \quad (63)$$

This implies that

$$t^{\max} \leq \min\{(\lambda + \mu + \nu)^j / a^{\vee j} \mid j = 1, \dots, r\}. \quad (64)$$

The bound (62) is always stronger than the bound (64), which is equivalent to stating that

$$\nu \cdot \theta \leq (\nu^j + \nu^{j+}) / a^{\vee j} \quad \text{for } j = 1, \dots, r. \quad (65)$$

We have checked (65) explicitly for all simple Lie algebras:  $A_r, B_r, \dots, F_4, G_2$ . Alternatively, one may regard the comparison of the bounds (62) and (64) as an indirect proof of (65), since from the point of view of the associated depth rules, the one leading to (62) is obviously the stronger one. We recall that  $A_r \simeq su(r+1)$ .

A weaker bound was previously found by Cummins for  $su(r+1)$ . As quoted in [7] it states that  $t^{\max} \leq \min\{S_1, S_r\}$ , and was obtained using symmetric function techniques. Here  $S_1$  and  $S_r$  are defined as in (17) and (34).

## 5 Comments

We anticipate that fusion multiplicities for higher rank  $su(r+1)$  may also be characterised by discretised polytope volumes. Whether or not such a volume may be measured straightforwardly is less clear. One could imagine that complications such as the integer-value considerations above would appear, and that they increase in complexity for higher rank.

We also believe that our approach may be extended to cover other simple Lie algebras, and intend to discuss that elsewhere. Since a generalisation of the BZ triangles to other Lie algebras is presently not known, our belief is mainly based on our alternative approach to the computation of fusion multiplicities. It relies on the depth rule and the relation to three-point functions in Wess-Zumino-Witten conformal field theory [16, 17]. In that work BZ triangles only appear as guidelines, while the basic building blocks are certain polynomials. It is an intriguing observation that the assignment of threshold levels to those polynomials is straightforward, whereas the derivation of (35) was quite cumbersome. One could therefore speculate that further progress, even for higher rank  $su(r+1)$ , is more likely to be made using the three-point function approach.

Along another line of generalisation, we are currently studying higher-point and higher-genus  $su(3)$  and  $su(4)$  fusions. Thus, those efforts are combining the work presented here with our recent results on higher-point and higher-genus  $su(2)$  fusion [5], and our general description of higher-point  $su(r+1)$  tensor product multiplicities [4].

Finally, when implemented in computer programs, we anticipate that our multiple sum formulas offer a computational advantage over more conventional methods such as Weyl group, and Young tableaux methods, for example.

### Acknowledgements

We thank P. Mathieu for commenting on the manuscript, and C. Woodward for informing us of reference [11]. The typing of this work was completed while J.R. visited the CRM, Montreal. He is grateful for its generous hospitality.

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